Macroeconomic models with physical and monetary dimension based on constrained dynamics

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▶ Motivation: the growth imperative
▶ Limits of economic models
▶ Modeling framework
▶ Model applications: exchange, production, monetary and biophysical stock-flow consistency
▶ Summary and conclusions
Motivation: Is economic growth imperative?

A variety of theories

1. Personal motivation
2. Interest, credit
3. Competition, profits
4. Technical progress
5. States & their institutions

Survey papers co-authored with Andreas Siemoneit, Berlin:


Definition of a growth imperative?

- Exterior conditions
  - Agent (meth. individualism): individual, entreprise, states
  - Increase economic effort
    - Work more, invest, R&D
  - Existential consequences
    - Loss of income/cost coverage, social exclusion

Or
Central role: technical change

- energy use explains Solow residual (Ayres et al., 2009; Kümmel, 2011)
- technical change allows for factor substitution
- vicious and virtuous circle of growth
- without growth: risk of unemployment, indebtedness, instability?
The social dilemma of economic growth

Productivity gains through automation
Additional resource use

Threat: loss of income
Threat: social security
Increase economic efforts
Growth policies
States

Individuals, entrepreneurs
Necessary ingredients of a model

- endogenous resource-driven technical change (Reiner Kümmel, LH14)
- investment, endogenous credit creation (Gaël Giraud, LH14)
- unemployment, income heterogeneity
- conspicuous consumption (no selfish utility functions)
- prisoner’s dilemma
- dynamics out of equilibrium
GENERAL EQUILIBRIUM MODELS

- market clearing
- stochastic shocks, but no uncertainty, instabilities, or coordination failures
- often lack of out-of-equilibrium foundations or multiple equilibria
- no dynamics of money & credit (neutrality)
- rational behavior
Constrained optimization of master utility function

- numerous constraints (budget, zero-profit equilibrium, ...)
- jump to the utility top ("tangent on pareto set"; Yves Bréchet, LH18)
- agents correctly anticipate all constraints
- ‘invisible hand’ creates no dilemmata
- aggregation has to be possible → representative agent assumption

- equilibrium models worthless if these conditions do not hold
- how do market forces act out of equilibrium?
**HISTORICAL ANALOGIES BETWEEN MECHANICS AND ECONOMICS**

Equilibrium was described in analogy to stationary states of mechanical systems.

<table>
<thead>
<tr>
<th>Year</th>
<th>Mechanics</th>
<th>Economics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1686, Newton</td>
<td>Dynamics</td>
<td></td>
</tr>
<tr>
<td>1788, Lagrange</td>
<td>Constrained Dynamics</td>
<td>Optimization</td>
</tr>
<tr>
<td>1838, Cournot</td>
<td></td>
<td>Optimization</td>
</tr>
<tr>
<td>1874, Early Neoclassicals / Walras</td>
<td></td>
<td>General Equilibrium</td>
</tr>
<tr>
<td>1954, Arrow/Debreu</td>
<td></td>
<td>GE as Optimization under Constraint</td>
</tr>
<tr>
<td>2018</td>
<td></td>
<td>Constrained Dynamics?</td>
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</tbody>
</table>
Role of Dynamics in Economic Models

- Dynamic equilibrium models describe a “quasi-static process”: the system is “at equilibrium at every point between its initial and final states” (Berry et al., 1978).

- For early neoclassicals, dynamics “did not mean intertemporal choices or equilibria but instead the adaptive processes that were thought to converge on the states analyzed in static theory” (Leijonhufvud, 2006, pp. 29–30).

- “very little has been done to address the unfinished business of the older neoclassical theory” (Leijonhufvud, 2006, pp. 29–30).
General Equilibrium: Inspired by Physics?

Lagrangian Mechanics:
- dynamics of interacting particles under constraints,
- conserved quantities such as energy.

⇒ Newton would have used Stock-Flow Consistent Agent-Based models.

⇒ Idea: Extend analogies between economics and mechanics:
⇒ from constrained optimization to constrained dynamics
IDEA: MOTION UNDER CONSTRAINT

model dynamics of stocks and flows (goods, financial assets, material, energy) and their restrictions (‘consistency’)

Miles Sabin, CC-BY-SA 2.0
### Relation Lagrangian Mechanics – Economics

<table>
<thead>
<tr>
<th>Mechanics</th>
<th>Economics</th>
</tr>
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<tbody>
<tr>
<td>velocity $v_j$</td>
<td>stocks, flows, prices $y_j$</td>
</tr>
<tr>
<td>(constraint) forces $f_i^j$</td>
<td>(constraint) forces $f_i^j$</td>
</tr>
<tr>
<td>mass $M_j$</td>
<td>economic power $\mu_i^j$:</td>
</tr>
<tr>
<td></td>
<td>ability to control a variable</td>
</tr>
</tbody>
</table>

\[
\dot{v}_j = \frac{1}{M_j} \sum_i f_i^j \\
\dot{y}_j = \sum_i \mu_i^j f_i^j
\]
Constraints restrict the phase space of the variables:

- individual budget constraints
- production functions (Leontief, Cobb-Douglas, LinEX)
- input–output consistency (Sandra Bouneau, LH18)
- monetary stock-flow consistency (Gaël Giraud, LH14+18): First law of financial economics
- energy conservation: First law of thermodynamics (Reiner Kümmel, LH14)
- mass conservation: First law of chemistry

In our model: Constraints generate constraint forces as in Lagrangian mechanics.
Behavioral Assumptions

Optimization / equilibrium models:
- Maximize master utility function under constraints

Our approach:
- different forces as desire to influence certain variables
- economic power is the ability to influence certain variables
- don’t jump to the top, but try to climb the mountain
- allow for heterogeneity among consumers and firms
- no equilibrium assumption, but possible convergence
DIFFERENTIAL-ALGEBRAIC EQUATION FRAMEWORK

$m$ agents
$n$ stocks $x_i$ & corresponding flows $\dot{x}_i$.

Identities / constraints:

$$Z_k(\vec{x}) = 0$$

(1)

Time evolution:

$$\ddot{x}_j(t) = \sum_{i=1}^{m} \mu^i_j f^j_i(\vec{x}, \vec{\dot{x}}) + \sum_{k=1}^{l} \lambda_k \frac{\partial Z_k(\vec{x}, \vec{\dot{x}})}{\partial \dot{x}_j}$$

(2)

$\mu^i_j$: economic power (ability of agent $j$ to influence flow $i$)
$f^j_i(\vec{x}, \vec{\dot{x}})$: economic force (wish of $i$ to influence flow $j$)

(Glötzl, Glötzl, Richters: Discussion Paper 2017)
Four short examples of work in progress

1. Investment and Saving
2. Exchange Model
3. Production Model
4. Monetary Stock-Flow Consistent Model
5. Monetary and Physical Flow Model
Example 2: Exchange Model

Constraints:

\[ \sum_{i} \dot{x}_i = 0 = Z_0. \] (3)

\[ m_i + p \dot{x}_i = 0 = Z_i \quad \forall \ i. \] (4)

Gradient climbing: Forces according to marginal utility:

\[ f_{x_i}^i \propto \frac{\partial U_i}{\partial x_i}, \quad f_{m_i}^i \propto \frac{\partial U_i}{\partial m_i}. \] (5)

Time evolution for \( x_i \):

\[ \dot{x}_i = \mu_i \frac{\partial U_i}{\partial x_i} + \lambda_0 + p \lambda_i. \] (6)

Slow auctioneer increases price of \( x \) slowly if supply > demand.
Edgeworth Box, Exchange Model

Exchange model with ‘slow’ tatonnement process.
EXAMPLE 3: 2X2 PRODUCTION MODEL

Constraints (for sector $i$: capital $K_i$, labor $L_i$, production $C_i$):

$$Z_i = C_i - K_i^{\kappa_i} L_i^{1-\kappa_i} = 0,$$
$$Z_K = \dot{K}_1 + \dot{K}_2 = 0,$$
$$Z_L = \dot{L}_1 + \dot{L}_2 = 0.$$

Firms increase profits given by:

$$\Pi_i = p_i C_i - r K_i - w L_i.$$

Household increases utility given by:

$$U = C_1^{\alpha_1} C_2^{\alpha_2}.$$

Time evolution (exemplary):

$$\dot{C}_1 = \mu_i \frac{\partial U}{\partial C_1} + \mu_f \frac{\partial \Pi_1}{\partial C_1} + \lambda_i.$$

Slow price adaptation:

$$\dot{r} = \mu' \sum_i \dot{K}_i^\top.$$
TIME EVOLUTION AND CONVERGENCE

[Diagram showing the evolution of various economic variables over time, with markers for conventional equilibrium and start points.]
Example 4: Stock-Flow Consistent Models

Households
Money $M^h$ balance

Government
Money $M^g$

\[ 0 = Z^h = Y - T - C - \dot{M}^h \]
\[ 0 = Z^g = G - T - \dot{M}^g \]
\[ \Delta M^h \text{ or } \dot{M}^h \text{ saving} \]
\[ \Delta M^g \text{ or } \dot{M}^g \text{ money emission} \]

Taxes
Income
Consumption
Production

\[ 0 = Z^f = C + G - Y \]

(Godley et al., 2007)

- constraints: consistency of stocks and flows
- behavioral functions and disequilibrium behavior
- discrete dynamical system, motion under constraint
Challenges with SFC models

Problems with these discrete time models

- $N$ variables, together with $K$ constraints
- (arbitrary) subset of $N - K$ behavioral functions can be chosen
- Example: consumption function: $C_t = c_y Y_D(t) + c_v M_{t-1}$

Our approach:

- behavioral forces for each variable
- needed: $K$ additional Lagrange multipliers
Household’s utility $U$ depends on consumption $C$ and money stock $M$.

Government spending $G$ is exogenous, $\theta$: tax rate. $Y = C + G$.

Constraint:

$$Z_0 = (1 - \theta)(C + G) - C - \dot{M} = 0.$$  \hspace{1cm} (7)

Time evolution:

$$\dot{C} = \mu \frac{\partial U}{\partial C} - \theta \lambda,$$  \hspace{1cm} (8)

$$\ddot{M} = \mu \frac{\partial U}{\partial M} - \lambda.$$  \hspace{1cm} (9)
RESULT OF THE REPRODUCTION

upper left: discrete original model.
others: continuous imitation with utility gradient climbing
Example 5: Ecological–Financial Model

Diagram:

- Households: Money M | Wealth V
- Production: W = Y = C
- Government: Wealth V_g | Money M
- Consumption: C
- Wages: W
- Dispersal income: Y_d
- Taxes: T
- Interest on money: i_H
- Government expenditure: G
- Harvest: biomass
- Sunlight as 'universal service'
- Atmosphere: heat emissions
- Stock of heat

Flows:
- Money
- Energy
- Goods
ECOLOGICAL–FINANCIAL MODEL

- demand-driven monetary SFC model, including interest-bearing debt
- flows and funds of energy (Georgescu-Roegen, 1971)
- ecosystem exhibits logistic growth
**Stability Analysis: Eco-Eco-Interaction**

![Graphs showing stability analysis](Barth et al., 2018)
CHARACTERISTICS OF THE FRAMEWORK

(1) incorporate behavioral assumptions different from optimization,
(2) relax macroscopic assumptions about aggregation of individual behavior,
(3) distinguish and model of *ex-ante* and *ex-post* dynamics,
(4) discuss slow price adaptation and out-of-equilibrium dynamics,
(5) treat stocks, flows, and their constraints consistently,
(6) formalize economic power, and
(7) include some well-known general equilibrium solutions as fixed points of the dynamical system.

Different economic theories can be represented within one single framework.
Work in progress:

- Constrained dynamics formalize economic forces, constraint forces, and power for economic models in and out of equilibrium.
- Models represent goods, production, money, energy, and materials consistently.
- To do: Combine and apply them.

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Example 3: Post-Keynesian Stock-Flow Consistent Model

Dynamical equations:

\[ C = c_y \cdot Y_D + c_v \cdot H(t-1), \]  
\[ Y_D = (1 - \theta)W \cdot N, \]  
\[ T = \theta \cdot W \cdot N, \]  
\[ \Delta H = G - T = Y_D - C. \]

\( Y_D \) disposable income, \( C \) consumption, \( H \) money stock of households, \( W \) wage per hour, \( N \) hours worked, \( T \) taxes, \( \theta \) tax rate, \( G \) government expenditures, consumption out of income (\( c_y \)) and wealth (\( c_v \)).
Assume a $U_h$ depends on consumption $C$ and money stock $H$. Government spending $G$ is exogenous. Constraint:

$$ Z_0 = 0 = (1 - \theta)(C + G) - C - \dot{H}. $$  \hspace{1cm} (14)

Time evolution:

$$ \dot{C} = \mu \frac{\partial U}{\partial C} - \theta \lambda_0, $$  \hspace{1cm} (15)

$$ \ddot{H} = \mu \frac{\partial U}{\partial H} - \lambda_0. $$  \hspace{1cm} (16)